# Hardness of discrepancy and related problems parameterized by the dimension

P. Giannopoulos, C. Knauer, M. Wahlström, D. Werner

March 22, 2010

#### Outline

Basics Our Results Hardness of Maximum-Empty-Subinterval Conclusion

#### Basics

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

#### Our Results

Overview

#### Hardness of Maximum-Empty-Subinterval

The construction Encoding vertices Encoding edges Correctness Approximation

#### Conclusion

Adaption to the other problems

・ 同 ト ・ ヨ ト ・ ヨ ト

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

소리가 소문가 소문가 소문가

3

#### How equally distributed is a point set?

$$D_O(P) := \left| \frac{|P \cap O|}{|P|} - \operatorname{vol}(O) \right|$$



Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

#### How equally distributed is a point set?

$$\mathcal{D}_O(P) := \left| \frac{|P \cap O|}{|P|} - \mathsf{vol}(O) \right|$$



Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

소리가 소문가 소문가 소문가

#### How equally distributed is a point set?



Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

#### How equally distributed is a point set?

$$\mathcal{D}_O(P) := \left| \frac{|P \cap O|}{|P|} - \mathsf{vol}(O) \right|$$



Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

The Star Discrepancy is defined as

$$D^*(P) = \max_{I \in \mathcal{I}^*} D_I(P)$$

where  $\mathcal{I}^\ast$  is the set of all boxes inside the unit cube that contain the origin.

The Box Discrepancy is defined as

$$D(P) = \max_{I \in \mathcal{I}} D_I(P)$$

where  $\ensuremath{\mathcal{I}}$  is the set of all boxes inside the unit cube.

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

## Fixed-parameter tractability and parameterized hardness

• A parameterized decision problem is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ .

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

## Fixed-parameter tractability and parameterized hardness

- A parameterized decision problem is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ .
- ► *L* is *fixed-parameter tractable*, if it can be decided in  $\mathcal{O}(f(k) \cdot |x|^c)$  time whether  $(x, k) \in L$ .

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

イロト イポト イヨト イヨト

## Fixed-parameter tractability and parameterized hardness

- A parameterized decision problem is a language  $L \subseteq \Sigma^* \times \mathbb{N}$ .
- ► *L* is *fixed-parameter tractable*, if it can be decided in  $\mathcal{O}(f(k) \cdot |x|^c)$  time whether  $(x, k) \in L$ .
- ► A problem is W[1]-hard if the k-CLIQUE problem can be reduced to it by a parameterized reduction.

Geometric Discrepancy Parameterized Complexity Maximum-Empty-Subinterval

## The Maximum-Empty-Subinterval problem

**Given:** A finite point set P inside the d-dimensional unit cube, a number V.

**Question:** Is there a box inside  $[0, 1]^d$  containing the origin and none of the points that has volume at least *V*?

The following problems are W[1]–hard with respect to the dimension (and NP–hard):

► MAXIMUM-EMPTY-SUBINTERVAL

・ロン ・回と ・ヨン ・ヨン

æ

The following problems are W[1]–hard with respect to the dimension (and NP–hard):

- ► Maximum-Empty-Subinterval
- Star-Discrepancy

3

The following problems are W[1]–hard with respect to the dimension (and NP–hard):

- ► Maximum-Empty-Subinterval
- Star-Discrepancy
- ► MAXIMUM-Empty-Box

The following problems are W[1]–hard with respect to the dimension (and NP–hard):

- ► Maximum-Empty-Subinterval
- Star-Discrepancy
- ► MAXIMUM-Empty-Box
- Box-Discrepancy

3

The following problems are W[1]–hard with respect to the dimension (and NP–hard):

► MAXIMUM-EMPTY-SUBINTERVAL

・ロン ・回と ・ヨン ・ヨン

æ

The construction Encoding vertices Encoding edges Correctness Approximation

イロン イヨン イヨン イヨン

æ

#### Overview

▶ Reduction from *k*−CLIQUE

The construction Encoding vertices Encoding edges Correctness Approximation

・ロン ・回と ・ヨン・

3



- ▶ Reduction from *k*−CLIQUE
- For G = ([n], E) and an integer k, we will construct a set of points in ℝ<sup>2k</sup> that admits an empty box of volume C<sup>k</sup> iff G has a k-clique.

The construction Encoding vertices Encoding edges Correctness Approximation

イロン イヨン イヨン イヨン



- ▶ Reduction from *k*−CLIQUE
- For G = ([n], E) and an integer k, we will construct a set of points in ℝ<sup>2k</sup> that admits an empty box of volume C<sup>k</sup> iff G has a k-clique.
- We will place points into k orthogonal planes to encode the vertices

The construction Encoding vertices Encoding edges Correctness Approximation

・ロン ・回と ・ヨン・



- ▶ Reduction from *k*−CLIQUE
- For G = ([n], E) and an integer k, we will construct a set of points in ℝ<sup>2k</sup> that admits an empty box of volume C<sup>k</sup> iff G has a k-clique.
- We will place points into k orthogonal planes to encode the vertices
- and additionally into their pairwise product to encode the edges.

The construction Encoding vertices Encoding edges Correctness Approximation



- ▶ Reduction from *k*−CLIQUE
- For G = ([n], E) and an integer k, we will construct a set of points in ℝ<sup>2k</sup> that admits an empty box of volume C<sup>k</sup> iff G has a k-clique.
- We will place points into k orthogonal planes to encode the vertices
- and additionally into their pairwise product to encode the edges.
- Observe: As the origin must be contained, the planes can be considered separately.

・ロン ・回 と ・ ヨ と ・ ヨ と

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロト ・回ト ・ヨト ・ヨト

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロン ・回と ・ヨン・

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロト ・回ト ・ヨト ・ヨト

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロト ・回ト ・ヨト ・ヨト

3

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロト ・回ト ・ヨト ・ヨト

 
 Outline Basics
 The construction Encoding vertices

 Our Results
 Encoding edges

 Hardness of Maximum-Empty-Subinterval Conclusion
 Correctness

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロン ・回と ・ヨン ・ヨン

 
 Outline Basics
 The construction Encoding vertices

 Our Results
 Encoding deges

 Hardness of Maximum-Empty-Subinterval Conclusion
 Correctness

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



・ロト ・回ト ・ヨト ・ヨト

 
 Outline Basics
 The construction

 Basics
 Encoding vertices

 Our Results
 Encoding edges

 Hardness of Maximum-Empty-Subinterval Conclusion
 Approximation

Let  $\mu > 1$  and  $C := 1/\mu^{n-1} < 1$ . In each of the k planes, we place n+1 points (n large rectangles) as follows.



► A selection of k such rectangles corresponds to a subset of vertices of G.

イロン イヨン イヨン イヨン

The construction Encoding vertices Encoding edges Correctness Approximation

#### How to forbid certain large rectangles?

The construction Encoding vertices Encoding edges Correctness Approximation

イロト イポト イヨト イヨト

#### How to forbid certain large rectangles?



The construction Encoding vertices Encoding edges Correctness Approximation

イロト イポト イヨト イヨト

#### How to forbid certain large rectangles?



The construction Encoding vertices Encoding edges Correctness Approximation

イロト イポト イヨト イヨト

#### How to forbid certain large rectangles?

We want to forbid boxes corresponding to u in the *i*-th  $\mathbb{R}^2$  and to v in the *j*-th  $\mathbb{R}^2$  for  $uv \notin E$ .



• Add a point in the product of the two planes  $(\mathbb{R}^4)$ .

The construction Encoding vertices Encoding edges Correctness Approximation

## How to forbid certain large rectangles?

We want to forbid boxes corresponding to u in the *i*-th  $\mathbb{R}^2$  and to v in the *j*-th  $\mathbb{R}^2$  for  $uv \notin E$ .



- Add a point in the product of the two planes ( $\mathbb{R}^4$ ).
- The two rectangles cannot be chosen at the same time.

The construction Encoding vertices Encoding edges Correctness Approximation

## How to forbid certain large rectangles?



- Add a point in the product of the two planes  $(\mathbb{R}^4)$ .
- The two rectangles cannot be chosen at the same time.
- ▶ Do this for all  $1 \le i \ne j \le k$  and all  $uv \notin E$ .

The construction Encoding vertices Encoding edges Correctness Approximation

## How to forbid certain large rectangles?



- Add a point in the product of the two planes  $(\mathbb{R}^4)$ .
- The two rectangles cannot be chosen at the same time.
- ▶ Do this for all  $1 \le i \ne j \le k$  and all  $uv \notin E$ .

The construction Encoding vertices Encoding edges Correctness Approximation

## How to forbid certain large rectangles?



- Add a point in the product of the two planes  $(\mathbb{R}^4)$ .
- The two rectangles cannot be chosen at the same time.
- ▶ Do this for all  $1 \le i \ne j \le k$  and all  $uv \notin \underline{E}$ .

The construction Encoding vertices Encoding edges Correctness Approximation



#### Lemma

G has a k-clique iff there is an empty box of size  $C^k$ .

#### Theorem

MAXIMUM-EMPTY-BOX is W[1]-hard with respect to the dimension.

#### Corollary

Unless W[1] = FPT, there is no algorithm running in time  $\mathcal{O}(f(d) \cdot |P|^c)$  for this problem.

The construction Encoding vertices Encoding edges Correctness Approximation

#### An even stronger result

► Observe: If there is no k-clique, we need to choose at least one rectangle of size at most C/µ in one plane.

The construction Encoding vertices Encoding edges Correctness Approximation

#### An even stronger result

- ► Observe: If there is no k-clique, we need to choose at least one rectangle of size at most C/µ in one plane.
- An empty box can have a total volume of at most  $C^k/\mu$ .

The construction Encoding vertices Encoding edges Correctness Approximation

#### An even stronger result

- ► Observe: If there is no k-clique, we need to choose at least one rectangle of size at most C/µ in one plane.
- An empty box can have a total volume of at most  $C^k/\mu$ .
- Choosing µ large creates a large gap between positive and negative instances.

The construction Encoding vertices Encoding edges Correctness Approximation

소리가 소문가 소문가 소문가

#### An even stronger result

- ► Observe: If there is no k-clique, we need to choose at least one rectangle of size at most C/µ in one plane.
- An empty box can have a total volume of at most  $C^k/\mu$ .
- Choosing µ large creates a large gap between positive and negative instances.
- Approximating the problem by, e. g., a factor of 1/2<sup>|P|</sup> is NP-hard!

Adaption to the other problems

・ロン ・回 と ・ ヨン ・ ヨン

3

## Shrink and lift

#### The proof can be modified to show the W[1]-hardness of

STAR-DISCREPANCY

Adaption to the other problems

・ロト ・回ト ・ヨト ・ヨト

æ

## Shrink and lift

The proof can be modified to show the W[1]-hardness of

- STAR-DISCREPANCY
- ► MAXIMUM-EMPTY-BOX

Adaption to the other problems

・ロン ・回と ・ヨン ・ヨン

æ

## Shrink and lift

The proof can be modified to show the W[1]-hardness of

- STAR-DISCREPANCY
- ► MAXIMUM-EMPTY-BOX
- ► BOX-DISCREPANCY.

Adaption to the other problems

・ロン ・回 と ・ヨン ・ヨン

Э

## Thank You.