On the computational complexity of Ham-Sandwich cuts, Helly sets and related problems

Christian Knauer (U Bayreuth) Hans Raj Tiwary (UL Bruxelles) Daniel Werner (FU Berlin)

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Basics

The Ham-Sandwich Theorem Our results *d*-Sum

d-Ham-Sandwich

The idea The construction Correctness Summary

Further results

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The Ham-Sandwich Theorem Our results *d*-Sum

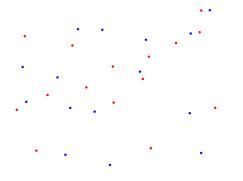
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The planar case

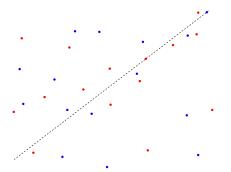
Let $P = R \cup B$. Then there is a line that *bisects* both sets simultaneously.



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The planar case

Let $P = R \cup B$. Then there is a line that *bisects* both sets simultaneously.



Such a line can be found in linear time! [Edelsbrunner, Waupotitsch; '86]

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General version

Theorem

For every d point sets in \mathbb{R}^d there exists a hyperplane that bisects them simultaneously.

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General version

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Proof: Borsuk-Ulam

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trivial algorithm: n^{d+1}

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General version

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For every d point sets in \mathbb{R}^d there exists a hyperplane that bisects them simultaneously.

Proof: Borsuk-Ulam

known bounds:

- trivial algorithm: n^{d+1}
- ▶ best known: $O(n^{d-1})$ [Lo, Matoušek, Steiger; '92]
- recently: O(n log^d n) for well separated point sets
 [Bárány, Hubard, Jéronimo; '08], [Steiger, Zhao; '09]

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The decision problem

Can we find a cut incrementally?

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The decision problem

Can we find a cut incrementally? (*d*-HAM-SANDWICH) Given: Sets P_1, \ldots, P_d in \mathbb{R}^d

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The decision problem

Can we find a cut incrementally?

(*d*-HAM-SANDWICH)

Given: Sets P_1, \ldots, P_d in \mathbb{R}^d **Question:** Is there a ham-sandwich cut through the origin?

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The decision problem

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Given: Sets P_1, \ldots, P_d in \mathbb{R}^d **Question:** Is there a ham-sandwich cut through the origin?

Alternatively:

Given: Sets P_1, \ldots, P_{d+1} in \mathbb{R}^d **Question:** Is there a ham-sandwich cut?

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No complexity results known so far.

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Our results

If the dimension is part of the input, d-HAM-SANDWICH is

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Our results

If the dimension is part of the input, d-HAM-SANDWICH is

▶ NP-hard (does not exclude *O*(*n*) for every fixed dimension)

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Our results

If the dimension is part of the input, d-HAM-SANDWICH is

- ▶ NP-hard (does not exclude O(n) for every fixed dimension)
- ▶ W[1]-hard when parameterized with the dimension

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Our results

If the dimension is part of the input, d-HAM-SANDWICH is

- ▶ NP-hard (does not exclude O(n) for every fixed dimension)
- W[1]-hard when parameterized with the dimension
- requires $n^{\Omega(d)}$ time, unless 3-SAT can be solved in $2^{o(n)}$

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The d-SUM problem

(*d*-SUM)

Given: A set of integers $S = \{s_1, \ldots, s_n\}$.

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The d-SUM problem

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Given: A set of integers $S = \{s_1, \ldots, s_n\}$. **Question:** Do *d* of them sum up to 0?

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Given: A set of integers $S = \{s_1, \ldots, s_n\}$. **Question:** Do *d* of them sum up to 0?

▶ parameterized version of SUBSET-SUM

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- W[1]-hard [Fellows, Koblitz; '93]

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- ▶ parameterized version of SUBSET-SUM
- W[1]-hard [Fellows, Koblitz; '93]
- requires n^{Ω(d)} time, unless 3-SAT can be solved in 2^{o(n)}
 [Pătrașcu, Williams; '10]

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The idea

Reduction from d-SUM

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The idea

Reduction from d-SUM

General idea: *Embed* the numbers as points into $\mathbb{R}^{f(d)}$ that have a certain property iff there are *d* numbers that sum up to 0.

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Reduction from d-SUM

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Here: Construct point sets P_1, \ldots, P_{d+1} in \mathbb{R}^{d+1} such that

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Reduction from d-SUM

General idea: *Embed* the numbers as points into $\mathbb{R}^{f(d)}$ that have a certain property iff there are *d* numbers that sum up to 0.

Here: Construct point sets P_1, \ldots, P_{d+1} in \mathbb{R}^{d+1} such that

there exists a linear ham-sandwich cut

 \Leftrightarrow

d of the numbers sum up to 0.

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Encoding the numbers

Let $S = \{s_1, \ldots, s_n\}$

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Encoding the numbers

Let $S = \{s_1, \dots, s_n\}$

Goal: Construct *d* sets P_1, \ldots, P_d in \mathbb{R}^{d+1} from *S* (and one extra set later)

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Goal: Construct d sets P_1, \ldots, P_d in \mathbb{R}^{d+1} from S (and one extra set later)

such that number appears in solution \Leftrightarrow linear cut goes through corresponding point.

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In dimension j: add point $p_i^j := \frac{1}{s_i} \cdot \mathbf{e}_j + \mathbf{e}_{d+1}$ for $1 \le i \le n$

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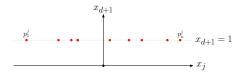
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Observe: if $h \cdot p_i^j = 0$ then $h_j = -h_{d+1}s_i$.

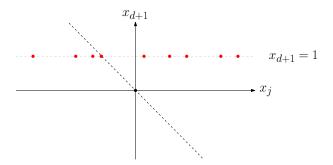
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Balancing points

Problem: Hyperplane through origin will not bisect the sets:

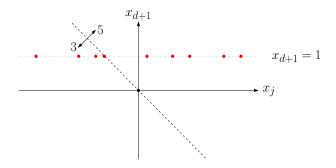


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Balancing points

Problem: Hyperplane through origin will not bisect the sets:



 \Rightarrow add *balancing* points

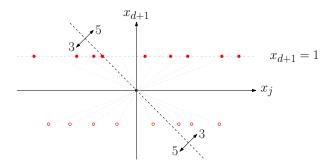
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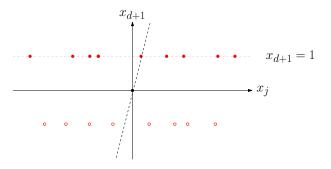
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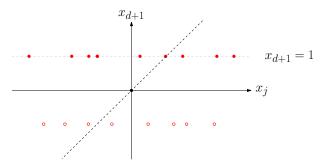
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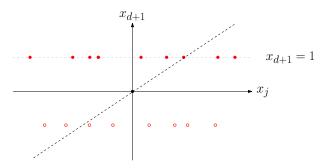
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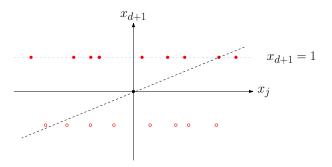
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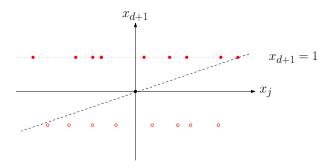
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Balancing points

Problem: Hyperplane through origin will not bisect the sets:



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The point q

One extra point will ensure that

none of the balancing points can lie on a linear cut

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The point q

One extra point will ensure that

- none of the balancing points can lie on a linear cut
- if points lie on linear cut \Rightarrow corresponding numbers sum to 0

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The point q

One extra point will ensure that

- none of the balancing points can lie on a linear cut
- if points lie on linear cut \Rightarrow corresponding numbers sum to 0

Set

$$q = -\sum_{i=1}^d \mathbf{e}_i$$

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and $P_{d+1} = \{q\}.$

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Some facts

Every linear cut

must contain q

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Some facts

Every linear cut

- must contain q
- contains exactly one point from each P_i

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Some facts

Every linear cut

- must contain q
- contains exactly one point from each P_i
- contains none of the balancing points

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Why it works

Claim:

There are d numbers that sum to 0.

 \Leftrightarrow

There is a linear ham-sandwich cut.

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Why it works

$$\Rightarrow$$
: Let $\sum_{j=1}^{d} s_{i_j} = 0$.

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

Let $h_j = s_{i_j}, \ 1 \le j \le d$

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

Let $h_j = s_{i_j}$, $1 \le j \le d$ and $h_{d+1} = -1$.

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

Let $h_j = s_{i_j}, \ 1 \le j \le d$ and $h_{d+1} = -1.$
Then

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

Let $h_j = s_{i_j}$, $1 \le j \le d$ and $h_{d+1} = -1$.
Then

$$hp_{i_j}^j = h\left(\frac{1}{s_{i_j}}\cdot\mathbf{e}_j + \mathbf{e}_{d+1}\right)$$

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

Let $h_j = s_{i_j}, \ 1 \le j \le d$ and $h_{d+1} = -1.$
Then

$$h p_{i_j}^j = h \left(rac{1}{s_{i_j}} \cdot \mathbf{e}_j + \mathbf{e}_{d+1}
ight) = s_{i_j} rac{1}{s_{i_j}} - 1$$

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Why it works

$$\Rightarrow$$
: Let $\sum_{j=1}^{d} s_{i_j} = 0$.
Let $h_j = s_{i_j}$, $1 \le j \le d$ and $h_{d+1} = -1$.
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Let $h_j = s_{i_j}, \ 1 \leq j \leq d$ and $h_{d+1} = -1.$

Then

$$hp_{i_j}^j=h\left(rac{1}{s_{i_j}}\cdot\mathbf{e}_j+\mathbf{e}_{d+1}
ight)=s_{i_j}rac{1}{s_{i_j}}-1=0,$$

so *h* halves each P_i , $1 \le i \le d$.

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Why it works

$$\Rightarrow: \text{Let } \sum_{j=1}^{d} s_{i_j} = 0.$$

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Further, as

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The idea The construction Correctness Summary

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$$hq = h\sum_{i=1}^{d} \mathbf{e}_i = \sum_{j=1}^{d} s_{i_j}$$

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q also lies on h.

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Why it works

 \Leftarrow : Let *h* be a linear cut.

The idea The construction Correctness Summary

Why it works

- \Leftarrow : Let *h* be a linear cut.
- Fact: h contains exactly one point from each P_i
 - (in particular, $h_{d+1} \neq 0$)

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Fact: each must be a point of the form $p_i^j = \frac{1}{s_i} \cdot \mathbf{e}_j + \mathbf{e}_{d+1}$

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Fact: each must be a point of the form $p_i^j = \frac{1}{s_i} \cdot \mathbf{e}_j + \mathbf{e}_{d+1}$

and wlog $h_{d+1} = -1$, thus $h_j = s_{i_j}$ for some i_j .

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Further, as q lies on h, we have

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$$0 = hq$$

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$$0 = hq = h\sum_{i=1}^{d} \mathbf{e}_i$$

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Why it works

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Fact: h contains exactly one point from each P_i

(in particular, $h_{d+1} \neq 0$)

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Fact: each must be a point of the form $p_i^j = \frac{1}{s_i} \cdot \mathbf{e}_j + \mathbf{e}_{d+1}$

and wlog $h_{d+1} = -1$, thus $h_j = s_{i_j}$ for some i_j .

$$0 = hq = h\sum_{i=1}^{d} \mathbf{e}_i = \sum_{j=1}^{d} h_j$$

The idea The construction Correctness Summary

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The idea The construction Correctness Summary

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What we have done

 $S = \{s_1, \ldots, s_n\}$ is a *d*-SUM instance

The idea The construction Correctness Summary

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$$S = \{s_1, \dots, s_n\}$$
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Further results

In a similar spirit one can show $n^{\Omega(d)}$ lower bounds for

Christian Knauer, Hans Raj Tiwary, Daniel Werner Ham-Sandwich cuts

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- ▶ more specific: Minimum Infeasible Subsystem for LP

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